

# Nearly tri-bimaximal mixing in the $S_3$ flavour symmetry

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**Abstract.** We present an analysis of the theoretical neutrino mixing matrix,  $V_{PMNS}^{th}$ , previously derived in the framework of the minimal  $S_3$ -invariant extension of the Standard Model. All entries in the neutrino mixing matrix,  $V_{PMNS}^{th}$ , the mixing angles and the Majorana phases are given as exact, explicit analytical functions of the mass ratios of the charged leptons and neutrinos, and one Dirac phase, in excellent agreement with the the latest experimental data. Here, it will be shown that all entries in  $V_{PMNS}^{th}$  are numerically very close to the tri-bimaximal form of the neutrino mixing matrix, so that  $V_{PMNS}^{th}$  may be written as  $V^{tri} + \Delta V_{PMNS}^{tri}$ . The small correction  $\Delta V_{PMNS}^{tri}$  is expressed as a sum of two terms: first, a small correction term proportional to  $m_e/m_\mu$  depending only on the charged lepton mass ratios and, second, a Cabbibo-like, small term,  $\delta t_{12}$ , which is a function of both the charged lepton and the neutrino mass ratios.

**Keywords:** Flavour symmetries, Quark and lepton masses and mixings, Neutrino masses and mixings

## INTRODUCTION

In the Standard Model (SM) the neutrinos are chiral and therefore massless, but in the past nine years the experiments and observations related to the neutrino physics showed us that neutrinos oscillate between states of well defined flavours. In this way it was established beyond reasonable doubt that neutrinos have non-vanishing masses [1].

The masses of the neutrinos are at least five orders of magnitude smaller than the electron mass, which is the lightest particle of all the fermions other than the neutrinos. The small magnitude of neutrino masses is naturally explained by assuming that the physical neutrinos are Majorana fermions which acquire their masses via the see-saw mechanism [1, 2].

The Standard Model explains almost all the experimental data at laboratory energies but it cannot give mass to the neutrinos and, even without the neutrino masses, it already has a large number of free parameters. In order to accommodate the massive neutrinos and at the same time reduce the number of free parameters in the theory, we formulated a minimal  $S_3$ -invariant extension of the Standard Model [3]. The flavour symmetry group of the extended theory is the group  $S_3$  of permutations of three objects, assumed to be unbroken at the Fermi scale.

The symmetry group  $S_3$  is the smallest non-Abelian group and it has only doublet and singlet irreducible representations. In order to have the theory invariant under  $S_3$ , we had to extend the concept of flavour and family to the Higgs sector, that is, in the extended form of the theory we have three Higgs  $SU(2)_L$  doublets that belong to one singlet and

the two components of the one doublet of  $S_3$ .

In this way, all the matter fields in the minimal  $S_3$ -invariant extension of the Standard Model, that is, quarks, leptons and Higgs fields, are in a reducible representation  $\mathbf{1}_s \oplus \mathbf{2}$  of  $S_3$ . We constructed the most general Yukawa Lagrangian and the Higgs potential invariant under this symmetry and obtained a generic form for the mass matrices of the Dirac fermions [3].

A further reduction of the number of parameters in the leptonic sector may be achieved by means of an Abelian  $Z_2$  symmetry. A set of charge assignments of  $Z_2$ , compatible with the experimental data on masses and mixings in the leptonic sector is given in [3]. This allowed us to reparametrize the mass matrices of the charged leptons and neutrinos, previously derived in [3], in terms of their eigenvalues and only one free parameter, the Dirac phase  $\delta$ . Then, we computed the neutrino mixing matrix,  $V_{PMNS}$ , and the neutrino mixing angles and Majorana phases as functions of the masses of charged leptons and neutrinos and only two undetermined phases. The numerical values of the reactor,  $\theta_{13}$ , and atmospheric,  $\theta_{23}$ , mixing angles are determined only by the masses of the charged leptons in very good agreement with experiment. The solar mixing angle,  $\theta_{12}$ , is almost insensitive to the values of the masses of the charged leptons, but its experimental value allowed us to fix the scale and origin of the neutrino mass spectrum [4].

More recently, we found exact, analytical expressions for the matrices of the Yukawa couplings in the leptonic sector expressed as functions of the masses of charged leptons and the vacuum expectation values of the Higgs bosons. With the help of the Yukawa matrices we computed the branching ratios of some selected FCNC processes [5], and more recently, the contribution of the exchange of neutral flavour changing scalars to the anomaly of the muon's magnetic moment [6]. We found that the interplay of the  $S_3 \times Z_2$  flavour symmetry and the strong mass hierarchy of charged leptons strongly suppress the FCNC processes in the leptonic sector, well below the experimental upper bounds by many orders of magnitude. The contribution of the FCNC to the anomaly,  $a_\mu$ , is at most 6% of the discrepancy between the experimental value and the Standard Model prediction for  $a_\mu$ , which is a small but non-negligible contribution [6].

In this short communication, we will show that all entries in  $V_{PMNS}^{th}$  derived in [4] are numerically very close to the tri-bimaximal form of the neutrino mixing matrix, so that,  $V_{PMNS}^{th}$  may be written as  $V^{tri} + \Delta V_{PMNS}^{tri}$ . The small correction  $\Delta V_{PMNS}^{tri}$  is expressed as a sum of two terms: first, a small correction term proportional to  $m_e/m_\mu$  depending only on the charged lepton mass ratios and, second, a Cabbibo-like, small term,  $\delta t_{12}$ , which is a function of both the charged lepton and the neutrino mass ratios.

## NEUTRINO MASSES AND MIXINGS

The neutrino mixing matrix  $V_{PMNS}$  is the product  $U_{eL}^\dagger U_\nu K$ , where  $K$  is the diagonal matrix of the Majorana phase factors, defined by

$$diag(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}) = K^\dagger diag(|m_{\nu_1}|, |m_{\nu_2}|, |m_{\nu_3}|) K^\dagger. \quad (1)$$

Except for an overall phase factor,  $e^{i\phi_1}$ , which can be ignored,  $K$  is

$$K = diag(1, e^{i\alpha}, e^{i\beta}). \quad (2)$$

The unitary matrix which diagonalizes the charged lepton mass matrix is of the form  $\mathbf{U}_{eL} = \Phi_e \mathbf{O}_{eL}$ , where  $\Phi_e$  is the diagonal phase matrix,  $\Phi_e = \text{diag}(1, 1, e^{i\delta_e})$ , and  $\mathbf{O}_{eL}$  is the orthogonal matrix

$$\mathbf{O}_{eL} \approx \begin{pmatrix} \frac{1}{\sqrt{2}}x \frac{(1+2\tilde{m}_\mu^2+4x^2+\tilde{m}_\mu^4+2\tilde{m}_e^2)}{\sqrt{1+\tilde{m}_\mu^2+5x^2-\tilde{m}_\mu^4-\tilde{m}_\mu^6+\tilde{m}_e^2+12x^4}} & -\frac{1}{\sqrt{2}} \frac{(1-2\tilde{m}_\mu^2+\tilde{m}_\mu^4-2\tilde{m}_e^2)}{\sqrt{1-4\tilde{m}_\mu^2+x^2+6\tilde{m}_\mu^4-4\tilde{m}_\mu^6-5\tilde{m}_e^2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}x \frac{(1+4x^2-\tilde{m}_\mu^4-2\tilde{m}_e^2)}{\sqrt{1+\tilde{m}_\mu^2+5x^2-\tilde{m}_\mu^4-\tilde{m}_\mu^6+\tilde{m}_e^2+12x^4}} & \frac{1}{\sqrt{2}} \frac{(1-2\tilde{m}_\mu^2+\tilde{m}_\mu^4)}{\sqrt{1-4\tilde{m}_\mu^2+x^2+6\tilde{m}_\mu^4-4\tilde{m}_\mu^6-5\tilde{m}_e^2}} & \frac{1}{\sqrt{2}} \\ -\frac{\sqrt{1+2x^2-\tilde{m}_\mu^2-\tilde{m}_e^2}(1+\tilde{m}_\mu^2+x^2-2\tilde{m}_e^2)}{\sqrt{1+\tilde{m}_\mu^2+5x^2-\tilde{m}_\mu^4-\tilde{m}_\mu^6+\tilde{m}_e^2+12x^4}} & -x \frac{(1+x^2-\tilde{m}_\mu^2-2\tilde{m}_e^2)\sqrt{1+2x^2-\tilde{m}_\mu^2-\tilde{m}_e^2}}{\sqrt{1-4\tilde{m}_\mu^2+x^2+6\tilde{m}_\mu^4-4\tilde{m}_\mu^6-5\tilde{m}_e^2}} & \frac{\sqrt{1+x^2}\tilde{m}_e\tilde{m}_\mu}{\sqrt{1+x^2-\tilde{m}_\mu^2}} \end{pmatrix}. \quad (3)$$

The Majorana neutrino mass matrix is obtained via the see-saw mechanism  $\mathbf{M}_\nu = \mathbf{M}_{\nu D} \tilde{\mathbf{M}}_R^{-1} (\mathbf{M}_{\nu D})^T$ , this matrix is diagonalized by a unitary matrix

$$U_\nu^T M_\nu U_\nu = \text{diag}(|m_{\nu_1}|e^{i\phi_1}, |m_{\nu_2}|e^{i\phi_2}, |m_{\nu_3}|e^{i\phi_3}), \quad (4)$$

where

$$U_\nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta_\nu} \end{pmatrix} \begin{pmatrix} \cos \eta & \sin \eta & 0 \\ 0 & 0 & 1 \\ -\sin \eta & \cos \eta & 0 \end{pmatrix}, \quad (5)$$

and

$$\sin^2 \eta = \frac{m_{\nu_3}-m_{\nu_1}}{m_{\nu_2}-m_{\nu_1}}, \quad \cos^2 \eta = \frac{m_{\nu_2}-m_{\nu_3}}{m_{\nu_2}-m_{\nu_1}}. \quad (6)$$

The resulting theoretical mixing matrix,  $V_{PMNS}^{th} = U_{eL}^\dagger U_\nu K$ , is given by

$$V_{PMNS}^{th} = \begin{pmatrix} O_{11} \cos \eta + O_{31} \sin \eta e^{i\delta} & O_{11} \sin \eta - O_{31} \cos \eta e^{i\delta} & -O_{21} \\ -O_{12} \cos \eta + O_{32} \sin \eta e^{i\delta} & -O_{12} \sin \eta - O_{32} \cos \eta e^{i\delta} & O_{22} \\ O_{13} \cos \eta - O_{33} \sin \eta e^{i\delta} & O_{13} \sin \eta + O_{33} \cos \eta e^{i\delta} & O_{23} \end{pmatrix} \times K. \quad (7)$$

To find the relation of our results with the neutrino mixing angles, we make use of the equality of the absolute values of the elements of  $V_{PMNS}^{th}$  and  $V_{PMNS}^{PDG}$  [8], that is

$$|V_{PMNS}^{th}| = |V_{PMNS}^{PDG}|. \quad (8)$$

This relation allowed us to derive expressions for the mixing angles in terms of the charged lepton and neutrino masses [4, 5].

The magnitudes of the reactor and atmospheric mixing angles,  $\theta_{13}$  and  $\theta_{23}$ , are determined by the masses of the charged leptons only. Keeping only terms of order  $(m_e^2/m_\mu^2)$  and  $(m_\mu/m_\tau)^4$ , we get

$$\sin \theta_{13} \approx \frac{1}{\sqrt{2}}x \frac{(1+4x^2-\tilde{m}_\mu^4)}{\sqrt{1+\tilde{m}_\mu^2+5x^2-\tilde{m}_\mu^4}} \quad \text{and} \quad \sin \theta_{23} \approx \frac{1}{\sqrt{2}} \frac{1+\frac{1}{4}x^2-2\tilde{m}_\mu^2+\tilde{m}_\mu^4}{\sqrt{1-4\tilde{m}_\mu^2+x^2+6\tilde{m}_\mu^4}}. \quad (9)$$

The magnitude of the solar angle depends on the charged lepton and neutrino masses, as well as the Dirac and Majorana phases

$$|\tan \theta_{12}|^2 = \frac{m_{\nu_2} - m_{\nu_3}}{m_{\nu_3} - m_{\nu_1}} \left( \frac{1 - 2 \frac{O_{11}}{O_{31}} \cos \delta \sqrt{\frac{m_{\nu_3} - m_{\nu_1}}{m_{\nu_2} - m_{\nu_3}}} + \left( \frac{O_{11}}{O_{31}} \right)^2 \frac{m_{\nu_3} - m_{\nu_1}}{m_{\nu_2} - m_{\nu_3}}}{1 + 2 \frac{O_{11}}{O_{31}} \cos \delta \sqrt{\frac{m_{\nu_2} - m_{\nu_3}}{m_{\nu_3} - m_{\nu_1}}} + \left( \frac{O_{11}}{O_{31}} \right)^2 \frac{m_{\nu_2} - m_{\nu_3}}{m_{\nu_3} - m_{\nu_1}}} \right). \quad (10)$$

From this expression, we found  $m_{\nu_3}$  as function of the solar angle, the squared neutrino mass differences and  $\phi_\nu$ , and, thus, fixed the origin and scale of the neutrino masses.

We are interested in the relation between our mixing matrix in Eq. (7) and the tri-bimaximal pattern [7]. In order to obtain this relation, it would be convenient to write  $\tan \theta_{12}$  as

$$\tan \theta_{12} = \frac{1}{\sqrt{2}} + \delta t_{12}, \quad (11)$$

where  $\delta t_{12}$  is a small quantity of the order of 6% of the tri-bimaximal value.

## DEVIATION OF THE MIXING MATRIX $\mathbf{V}_{PMNS}^{th}$ FROM THE TRI-BIMAXIMAL FORM

The previous results on neutrino masses and mixings weakly depend on the Dirac phase  $\delta$ , for simplicity we will assume in this work that  $\delta = \pi/2$ . From the expression (7), and the Eq. (11), we may write the mixing matrix as follows,

$$\mathbf{V}_{PMNS}^{th} = \mathbf{V}_{PMNS}^{tri} + \Delta \mathbf{V}_{PMNS}^{tri}, \quad (12)$$

where the tri-bimaximal form  $\mathbf{V}_{PMNS}^{tri}$  [7] is

$$\mathbf{V}_{PMNS}^{tri} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}, \quad (13)$$

and

$$\Delta \mathbf{V}_{PMNS}^{tri} = \Delta \mathbf{V}_e + \delta t_{12} \frac{(\sqrt{2} + \delta t_{12})}{g(\delta t_{12})} \Delta \mathbf{V}_\nu, \quad (14)$$

where

$$g(\delta t_{12}) = 1 + \frac{2}{3} \delta t_{12} (\sqrt{2} + \delta t_{12}). \quad (15)$$

Comparing the expression (7) for the neutrino mixing matrix, with the tri-bimaximal form (13) we get,

$$\Delta \mathbf{V}_e \approx \begin{pmatrix} -\frac{2}{3} \frac{s_{13}^2}{1+c_{13}} & -\frac{1}{3} \frac{s_{13}^2}{1+c_{13}} & s_{13} \\ \frac{5}{2\sqrt{6}} \frac{x^2}{1+\sqrt{1+\frac{5}{2}x^2}} & \frac{1}{4} \sqrt{\frac{1}{3}} \frac{x^2}{1+\sqrt{1+\frac{1}{4}x^2}} & -\frac{1}{2\sqrt{2}} \frac{x^2}{\sqrt{1-\tilde{m}_\mu^2+x^2}} \\ \sqrt{\frac{1}{6}} \frac{x^2}{1+\sqrt{1+x^2}} & \frac{1}{4} \sqrt{\frac{1}{3}} \frac{x^2}{1+\sqrt{1+\frac{1}{4}x^2}} & 0 \end{pmatrix}, \quad (16)$$

where  $x = m_e/m_\mu$ ,  $\tilde{m}_\mu = m_\mu/m_\tau$  and  $s_{13} \approx 1/\sqrt{2}x(1+4x^2-\tilde{m}_\mu^4)/\sqrt{1+\tilde{m}_\mu^2+5x^2-\tilde{m}_\mu^4}$ .

Notice that all entries in Eq. (16) are proportional to  $x^2$  except for the  $(\Delta \mathbf{V}_e)_{13}$  which is proportional to  $x$ . Therefore, in the limit of vanishing electron mass,  $\Delta \mathbf{V}_e \rightarrow 0$ .

The matrix  $\Delta \mathbf{V}_\nu$  can be written as

$$\Delta \mathbf{V}_\nu = \begin{pmatrix} -\left(\frac{2}{3}\right)^{3/2} \frac{1-\frac{s_{13}^2}{1+c_{13}}}{1+\sqrt{1-\frac{2}{3}\frac{\delta t_{12}(\sqrt{2}+\delta t_{12})}{g(\delta t_{12})}}} & \left(\frac{1}{3}\right)^{3/2} \frac{1-\frac{s_{13}^2}{1+c_{13}}}{1+\sqrt{1+\frac{1}{3}\frac{\delta t_{12}(\sqrt{2}+\delta t_{12})}{g(\delta t_{12})}}} & 0 \\ \left(\frac{2}{3}\right)^{3/2} \frac{1-2x^2}{\sqrt{1+\frac{5}{2}x^2}\left(1+\sqrt{1+\frac{4}{3}\frac{\delta t_{12}(\sqrt{2}+\delta t_{12})}{g(\delta t_{12})}\frac{1-2x^2}{1+\frac{5}{2}x^2}}\right)} & -\frac{2}{3\sqrt{3}} \frac{1-2x^2}{\sqrt{1+\frac{1}{4}x^2}\left(1+\sqrt{1-\frac{2}{3}\frac{\delta t_{12}(\sqrt{2}+\delta t_{12})}{g(\delta t_{12})}\frac{1-2x^2}{1+\frac{1}{4}x^2}}\right)} & 0 \\ \left(\frac{2}{3}\right)^{3/2} \frac{1+\frac{1}{4}x^2}{\sqrt{1+x^2}\left(1+\sqrt{1+\frac{4}{3}\frac{\delta t_{12}(\sqrt{2}+\delta t_{12})}{g(\delta t_{12})}\frac{1+\frac{1}{4}x^2}{1+x^2}}\right)} & -\frac{2}{3\sqrt{3}} \frac{1+x^2}{\sqrt{1+\frac{1}{4}x^2}\left(1+\sqrt{1+\frac{2}{3}\frac{\delta t_{12}(\sqrt{2}+\delta t_{12})}{g(\delta t_{12})}\frac{1+x^2}{1+\frac{1}{4}x^2}}\right)} & 0 \end{pmatrix} \quad (17)$$

notice that all the entries in the third column of  $\Delta \mathbf{V}_\nu$  vanish.

From Eq. (14) and Eq. (16) we can see that in the limit of  $m_e = 0$  and  $\delta t_{12} = 0$  the deviation from the tri-bimaximal pattern is exactly zero, that is

$$\Delta \mathbf{V}_{PMNS}^{tri} = 0. \quad (18)$$

The value for  $\delta t_{12}$  fixes the scale and the origin of the neutrino masses. If we take for  $\delta t_{12}$  the experimental central value  $\delta t_{12} \approx -0.04$ , we obtain [5]

$$\begin{aligned} |m_{\nu_2}| &\approx 0.056 eV \\ |m_{\nu_1}| &\approx 0.055 eV, \end{aligned}$$

and

$$|m_{\nu_3}| \approx 0.022 eV.$$

When we take for  $\delta t_{12}$  the tri-bimaximal value  $\delta t_{12} = 0$ , the neutrino masses are

$$m_{\nu_1} = 0.0521 eV \quad m_{\nu_2} = 0.0528 eV \quad \text{and} \quad m_{\nu_3} = 0.0178 eV \quad (19)$$

In both cases the  $S_3$  invariant extension of the SM predicts an inverted hierarchy. Since the tri-bimaximal value of  $\delta t_{12}$  differs from the experimental central value by less than 6% of  $\tan \theta_{12}$ , the difference in the corresponding numerical values of the neutrino masses are not significative within the present experimental uncertainties.

## CONCLUSIONS

In the minimal  $S_3$ -invariant extension of the SM, the flavour symmetry group  $S_3 \times Z_2$  relates the mass spectrum and mixings, and reduces the number of free parameters in the leptonic sector of the theory. This allowed us to predict two mixing angles,  $\theta_{13}$  and  $\theta_{23}$ , as function of the charged lepton masses only, in excellent agreement with the experimental values, while the other neutrino mixing angle,  $\theta_{12}$ , depends also on the neutrino mass spectrum. The tangent of the solar angle,  $\tan \theta_{12}$ , fixes the scale and origin of the neutrino masses which has an inverted hierarchy. In this model, we found that the deviation of the theoretical neutrino mixing matrix,  $V_{PMNS}^{th}$ , from the tri-bimaximal pattern is very small. In this form of the theory, the flavour violating processes are strongly suppressed by the flavour symmetry  $S_3 \times Z_2$  and the strong mass hierarchy of the charged leptons. Processes that proceed through the exchange of flavour changing neutral currents give information on the vacuum expectation values for the scalar Higgs bosons. The contribution of FCNC to the magnetic moment anomaly of the muon is at most 6% of the discrepancy between the experimental value and the Standard Model prediction, this contribution is small but non-negligible [6].

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